

The clustering of extreme values for some asymmetric GARCH-type models

Il clustering di valori estremi per alcuni modelli della famiglia GARCH con asimmetria

Fabrizio Laurini

Abstract Several models with conditional heteroskedasticity have been studied in financial econometrics, with the simple GARCH(1,1) with Gaussian innovation representing the standard benchmark. There is evidence of asymmetry in some daily data and more flexible models, which take such an asymmetry into account, have become recently popular. Understanding the extremal behaviour of asymmetric processes becomes very important to build proper inference about extremal events. For processes satisfying mild mixing conditions the clustering of extreme values is characterized by a single key-parameter, known as the extremal index, which represents the average clusters size of values which exceed a high-level threshold. An approach extending results for the GARCH(1,1) is presented, with skew- t innovation.

Abstract In finanza econometrica si sono studiati diversi modelli condizionatamente eteroschedastici, tra cui il semplice GARCH(1,1) rappresenta lo standard di riferimento. In alcuni dati finanziari, con cadenza giornaliera, vi è evidenza di asimmetria nella distribuzione marginale e modelli flessibili che tengono in considerazione tale aspetto sono divenuti popolari. Comprendere le proprietà di questi modelli, quando si considerano gli avvenimenti estremi, è di importanza cruciale per sviluppare un'opportuna inferenza. Per processi che soddisfano certe condizioni di regolarità, la tendenza a formare gruppi di valori estremi è caratterizzata da un singolo parametro-chiave, noto come l'indice estremo. Quest'ultimo esprime in funzione della dimensione media di un gruppo di eventi estremi oltre una certa soglia. Si presenta un risultato per il modello GARCH(1,1) con innovazione skew- t .

Key words: Asymmetric distributions, Extremal index, GARCH

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1 Introduction

Consider the daily log-returns $X_t = \log P_t - \log P_{t-1}$, where P_t , $t = 1, 2, \dots$, is the price of a generic asset. Then a broad class of models, mostly adopted to describe the market volatility, is the generalized autoregressive conditionally heteroscedastic (GARCH). For market risk management one of the most important issue is the presence of extreme values of daily log-returns. Therefore, understanding the extreme properties for such processes is fundamental, and this can be achieved by considering the marginal and the clustering properties of GARCH processes.

GARCH(p, q) processes, for integers p and q , and parameters $\lambda_i, i = 1, \dots, q$ and $\beta_j, j = 1, \dots, p$ which satisfy stationarity, have the form

$$X_t = \sigma_t Z_t \quad \text{with} \quad \sigma_t^2 = \alpha_0 + \sum_{i=1}^q \lambda_i X_{t-i}^2 + \sum_{j=1}^p \beta_j \sigma_{t-j}^2, \quad t = 1, 2, \dots, \quad \text{and} \quad \alpha_0 > 0. \quad (1)$$

For fixed t Z_t and σ_t are independent. In the sequel, the innovation $\{Z_t\}$ will be assumed strict white noise (SWN).

GARCH(p, q) processes $\{X_t\}$ have mixing properties, so that the key parameter for quantifying the impact of extreme values is the extremal index $\theta_X \in (0, 1]$. The extremal index θ_X measures the level of clustering of extreme values, with the clustering of extreme increasing for θ_X decreasing. The special case $\theta_X = 1$ means no clustering of extremes. Another interpretation of the extremal index is provided using the cluster size distribution $\pi_X(i), i = 1, 2, \dots$, since $\sum_{i=1}^{\infty} i \pi_X(i) = (\theta_X)^{-1}$, so θ_X is the reciprocal of the limiting mean cluster size of extreme values.

When $\{Z_t\}$ is symmetric the extreme properties of processes (1) have been studied by Basrak *et al.* (2002), which discuss extreme properties for the squared of the GARCH(p, q) process. For the special case of GARCH(1, 1) processes, results for θ_X are available (see Laurini and Tawn, 2012) but generalization to higher-order are still under investigation.

Nevertheless, there are studies in the financial and econometric literature which support asymmetry in the log-return process. To accommodate such empirical finding it is necessary to relax the assumption of symmetric marginal distribution of Z (provided some moments conditions are satisfied). Even for the Exponential GARCH, which allows for asymmetric response on the volatility $\{\sigma_t\}$, the $\{X_t\}$ process is still symmetric about zero, provided that Z is a symmetric zero-mean random variable.

We provide a Monte Carlo algorithm for the numerical evaluation of θ_X , and the associated cluster size distribution $\pi_X(\cdot)$. We first obtain the cluster size distribution $\pi_{X^2}(\cdot)$ and the extremal index θ_{X^2} for the square of the process, and then derive the desired quantities with proper thinning of the squared process. For sake of simplicity we discuss details only for GARCH(1, 1) with Z following a skew- t innovation (with degree of freedom such that the variance exists), and contrast results with other asymmetric models.

The outline of the paper is the following: in Section 2 we list the class of GARCH-type models that can be considered and introduces all restrictions and conditions. Section 3 provides an overview of the tail chain properties. Section 4 gives results of non-symmetric thinning. Further comments will be provided in Section 5.

2 Models, conditions and technicalities

Let us focus on the simpler form of model (1), with $p = q = 1$ and Z_t as SWN following a skew- t . The representation of skew- t requires some preliminaries. For a skew-normal random variable Y , a natural form of skew- t is given by the transform $Z = Y / \sqrt{(\chi_v^2)/v}$, that has density

$$f_Z(z) = 2t_v(z)T_{v+1}(w(z)), \quad \text{with} \quad w(z) = \alpha z \sqrt{\frac{v+1}{v+z^2}},$$

with t_v and T_v representing, respectively, the density and distribution of a Student- t with v degree of freedoms. The parameter α is the shape parameter that regulate the skewness. When $\alpha = 0$ we have symmetric Z , while with $\alpha > 0$ we have positively skew distribution. We rescale Z so that it is a zero-mean unit-variance random variable. Further details on skew-normal and other generalizations are given in Azzalini and Capitanio (2003).

Now, consider the quantity

$$C_t = \lambda Z_{t-1}^2 + \beta, \tag{2}$$

then, for stationarity of this process, λ and β must satisfy the constraint $E\{\log(C_t)\} < 0$, and for finite variance $\lambda + \beta < 1$.

The function $E(C^\zeta)$ is continuous and convex in ζ , with $\zeta = \kappa$ being the unique solution of the equation $E(C^\zeta) = 1$ for $\zeta > 0$. Then, under our assumptions about $\{Z_t\}$, κ satisfies

$$\int_{-\infty}^{\infty} (\lambda z^2 + \beta)^\kappa f_Z(z) dz = 1, \tag{3}$$

where $f_Z(z)$ is the marginal density of the iid sequence $\{Z_t\}$. Therefore, κ can be found by numerical solution of equation (3) is trivial and solution to find $\kappa > 0$ is immediate as the function is convex with one root being zero. As a function of ζ , we plot $E(C^\zeta)$ in Figure 1 for $\lambda = 0.11$ and $\gamma = 0.88$ for a variety of Z_t .

Given the value of κ one result of Mikosch and Starica (2000) is that σ_t and $|X_t|$ are both regularly varying random variables with index 2κ , i.e.

$$\Pr(\sigma_t > x) \sim \delta_0 x^{-2\kappa} \quad \text{and} \quad \Pr(|X_t| > x) \sim E(|Z_t|^{2\kappa}) \Pr(\sigma_t > x) \quad \text{for } x \rightarrow \infty,$$

where δ_0 is a positive constant.

The root κ does not depend on the sign of the shape parameter, due to the squared power into (3). There is a non-trivial link between the clustering of extremes and the value of κ . Here, we simply highlight that the marginal distribution of X has heavier tail when Z is Skew- t .

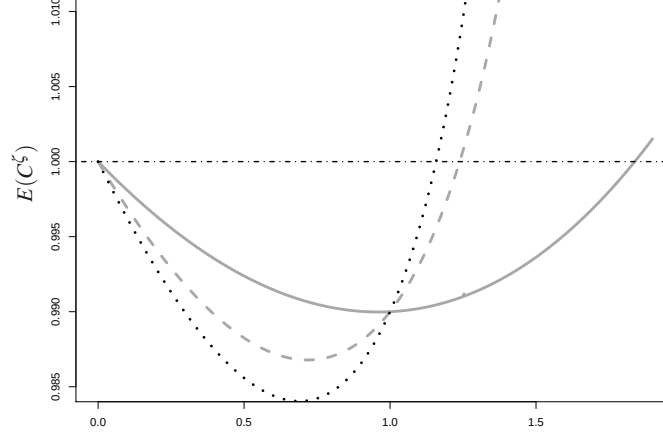


Fig. 1 Plot of $E(C^\zeta)$ for $\lambda = 0.11$ and $\gamma = 0.88$ and a variety of choices for Z . The solid grey is for Z standard Gaussian, the dashed grey is for Z Student- t and the black dotted is for Z Skew- t with (absolute value of the) shape set equal to 3. The sign of the shape parameter do not affect the value of $E(C^\zeta)$. The root κ is found when $E(C^\zeta) = 1$, which is represented by the horizontal black thin line (dotted and dashed).

For the given value of κ we then want to consider $|\cdot|$ be any norm, and define $\mathfrak{K} = \{\mathbf{x} \in \mathbb{R}^{2h} : |\mathbf{x}| = 1\}$. Then Mikosch and Starica (2000) show that for all h the vector $\mathbf{Y}_h^{(2)} = (X_1^2, \sigma_1^2, \dots, X_h^2, \sigma_h^2)$ exhibits a joint tail behaviour corresponding to multivariate regular variation with index κ , i.e. for any Borel set $B \subseteq \mathfrak{K}$

$$\frac{\Pr(|\mathbf{Y}_h^{(2)}| > rq, \mathbf{Y}_h^{(2)} / |\mathbf{Y}_h^{(2)}| \in B)}{\Pr(|\mathbf{Y}_h^{(2)}| > q)} \rightarrow r^{-\kappa} \Pr\{\Omega \in B\} \quad \text{for } r > 0, q \rightarrow \infty, \quad (4)$$

and with Ω being an angular random variable on the space, of dimension $2h - 1$, defined by this norm. The distribution of Ω , termed the spectral measure, is given by

$$\Pr\{\Omega \in \cdot\} = \frac{E \left\{ |\mathbf{Z}_h^{(2)}|^v I_{\{|\mathbf{Z}_h^{(2)}|/|\mathbf{Z}_h^{(2)}| \in \cdot\}} \right\}}{E \left\{ |\mathbf{Z}_h^{(2)}|^v \right\}}, \quad (5)$$

where I is the indicator function, and $\mathbf{Z}_h^{(2)} = \left((Z_1^2, 1), C_2(Z_2^2, 1), \dots, \{\prod_{i=2}^h C_i\}(Z_h^2, 1) \right)$. For our purposes, selecting $h = 1$ will suffice.

3 Conditional simulation of extremes

Very often it may be desirable simulating directly a cluster of exceedances instead of simulating the whole time series and extracting clusters of values above a threshold.

The technique we are going to set up comes from the study of bivariate extremes, and we use the $\mathcal{L}^1$ norm, thus $|\mathbf{x}| = \sum |x_i|$.

We have seen that expression (4) is the product of independent radial and angular pseudo-polar components, where the radial is $x^{-\kappa}$ and the angular is $\Pr\{\Omega \in \cdot\}$.

Our aim is to simulate contemporaneously a couple (X_1^2, σ_1^2) but, in practice we can concentrate on the vector $\mathbf{Z}_h^{(2)}$. To understand this argument observe that $(X_t^2, \sigma_t^2) = \sigma_t^2(Z_t^2, 1)$, so we focus on the pair $(Z_t^2, 1)$. Due to the connections with bivariate extremes we also need to define the pseudo-polar coordinates as $R = 1 + z$ and $\Omega = \frac{1}{1+z}$. Pseudo-polar coordinates are useful because, in bivariate extremes, the limit process factorizes across radial and angular components, which are independent each other. This property was used in Laurini and Tawn (2012). After simulating from the radial and angular component we map the original (X_t^2, σ_t^2) through the transformations $X_t^2 = R(1 - \Omega)$ and $\sigma_t^2 = R\Omega$.

The simulation of (4) is simple due to the independence. The radial component can be simulated directly from inversion. On the contrary, a Monte Carlo technique is required from simulating the angular component in (5), and a subsequent empirical inversion is required.

To build the tail chain, from (X_t^2, σ_t^2) we keep only those X_t^2 sufficiently large, and we use again regular variation. Since

$$\frac{\Pr(R > xq)}{\Pr(R > q)} = x^{-\kappa},$$

then it can be thought as the conditional probability $\Pr(R > xq | R > q)$, for $x > q$; for this reason we can argue that sampling from R correspond to sample from the conditional distribution

$$F_{\frac{R}{q} > 1}(x; \nu) = \Pr\left\{ \frac{R}{q} > x \mid \frac{R}{q} > 1 \right\} = x^{-\kappa}.$$

This argument suggests to adopt $q = 1$ as threshold for X_t^2 ; values in the (X_t^2, σ_t^2) space for which $X_t^2 \leq 1$ will be then discarded. Some improvement can be achieved if we could directly simulate from the distribution of R and Ω such that $X_t^2 = (1 - \Omega)R > 1$.

4 Thinning of the squared process

We can move from the cluster size distribution of the square of a GARCH(1, 1), to the cluster size distribution π_i , of the corresponding GARCH(1, 1) process. We need to rule out observations of the square GARCH that do not appear in the GARCH. More precisely, we denote with $\gamma = 1/\Pr(Z > 0)$, so that for $\alpha = 0$ $\gamma = 2$, and using $\Pi(\cdot)$ as the probability generating function of the cluster size probability, we have that, for the limiting squared time-normalized process of exceedances $N^{(2)}(0, 1)$,

$$\Pr(\max(X_1^2, \dots, X_n^2) < u_n) \rightarrow \sum_{k=0}^{\infty} \Pr(N^{(2)}(0, 1) = k) \gamma^{-k}.$$

By using conditional arguments, as in Embrechts et al. (1997), we can rewrite such limiting form, using the above thinning as

$$\pi_i = \left(1 - \Pi(1/\gamma)\right)^{-1} \sum_{m=i}^{\infty} \binom{m}{m-i} \pi_m^{(2)} \gamma^{-m}, \quad i = 1, 2, \dots$$

5 Final remarks

The dependence of extreme values involving θ_X is rather complicated, when λ and β are fixed. The dependence on α , from which γ is computed, makes the extreme clustering slightly weaker. For fixed λ , θ_X decreases monotonically with increasing β . Similarly, for fixed β , θ_X decreases monotonically with increasing λ . Though θ_X changes in a largely consistent way with v , with smaller θ_X occurring for smaller v , this property breaks down when $\lambda + \beta \approx 1$, i.e. when the process is close to non-stationary.

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References

1. Azzalini, A., Capitanio, A.: Distributions generated by perturbation of symmetry with emphasis on a multivariate skew t-distribution. *J. Royal Stat Soc B*, **61**, 579–602 (2003)
2. Basrak, B., Davis, R. A., Mikosch, T.: Regular variation of GARCH processes. *Stoch Proc Applicat*, **99**, 95–115 (2002)
3. Embrechts, P., Klüppelberg, C. and Mikosch, T.: *Modelling Extremal Events: for Insurance and Finance*. Heidelberg: Springer-Verlag (1997)
4. Laurini, F., Tawn, J.A.: The extremal index for GARCH(1, 1) processes. *Extremes*, **15**, 511–529 (2012)
5. Mikosch T., Stărică C.: Limit theory for the sample autocorrelations and extremes of a GARCH(1,1) process. *Ann Stat*, **28**, 1427–1451 (2000)